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SHORTER COMMUNICATIONS

APPLICATION OF THE HEAT-BALANCE INTEGRAL TO MELTING PROBLEMS WITH INITIAL SUBCOOLING

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NOMENCLATURE

B_1, constant defined by equations (8a);
B_2, constant defined by equations (21) and (22);
c, specific heat;
D_1, constant defined by equation (4);
D_2, constant defined by equation (4);
F, prescribed heat flux at $x = 0$;
G, function defined by equations (16);
h, heat-transfer coefficient at $x = 0$;
H, position of melting interface;
k, thermal conductivity;
k_{sl} k_s/k_l, ratio of solid to liquid thermal conductivity;
L, latent heat of fusion;
R, $\theta_m/(\theta_w - \theta_m)$, dimensionless subcooling ratio;
St, $\theta_w - \theta_m$, Stefan number;
t, time coordinate;
t_0, time when melting begins;
T, temperature;
T_i, initial temperature;
T_m, melting temperature;
T_∞, ambient temperature;
x, spacial coordinate.

Greek symbols

α, thermal diffusivity;
α_s, α_s/α_l ratio of solid to liquid thermal diffusivity;
β, hH/k_l dimensionless melting thickness;
δ, penetration depth into the solid region;
θ, $C_l(T - T_i)/L$, dimensionless temperature;
θ_m, dimensionless melting temperature;
θ_w, $\theta(0)$, dimensionless wall temperature;
θ_∞, dimensionless ambient temperature;
λ, $h\delta/k_l$, dimensionless penetration depth;
ρ, density;
σ, $FH/\rho L\alpha_l$, dimensionless melting thickness;
τ, dimensionless time defined by equation (12);
ϕ, $F\delta/\rho L\alpha_l$, dimensionless penetration depth;
ψ, melting constant defined by equations (7) and (8);
ω, dimensionless time defined by equation (16).

Subscripts

l, liquid;
s, solid.

INTRODUCTION

MANY practical heat-transfer problems involve a change of phase of the material due to either melting or freezing. Most of the existing work [1–5], however, deals only with situations in which the medium is initially at its melting temperature. As noted in a recent publication [5], this situation is not only unrealistic for most practical applications, but is also difficult to attain even under controlled laboratory conditions.

One of the most powerful solution techniques for the phase change problem is the heat-balance integral method [3–4]. In this method, the heat-conduction equation is approximated by an overall energy balance at regions of interest in both phases. With an assumed temperature profile and in conjunction with the energy-balance condition at the phase front, a system of differential equations are obtained. Approximate solutions for the interface location and other important physical parameters are then generated. For problems with zero subcooling, results [3] show that this 'pure integral' method is quite effective in generating reliable approximate solutions. But for problems with initial subcooling, the method yields a system of non-linear differential equations [4]. Solution to these equations can be just as complicated as a direct numerical solutions for the exact result. The heat-balance integral method, therefore, has yet been considered as an effective approximation method for melting problems with initial subcooling.

The objective of this work is to show that if the energy-balance condition at the phase front is expressed by a 'collocation method', the integral technique can be applied effectively to melting problems with initial subcooling. Developed originally by Goodman [3] to overcome the non-linearity encountered in problems with zero-subcooling, the 'collocation method' replaces the traditional energy-balance condition at the phase front by two relations between the medium's temperature profiles at both phases and their derivatives. Unlike the 'pure integral' technique, approximate solutions for the medium's temperature and the phase front location for problems with initial subcooling can be obtained either in closed form or as a numerical quadrature. To demonstrate the utility of the present method, detailed solutions for the melting of a one-dimensional semi-infinite subcooled plane subjected to three common types of boundary conditions are presented. The accuracy of these solutions is demonstrated by comparison with the available exact solution and numerical results. Based on these approximate solutions, the general effect of subcooling on the physics of melting is discussed.

MATHEMATICAL FORMULATION

Consider the idealized problem of the melting of a semi-infinite solid initially at a uniform temperature T_i ~~less than~~ its melting temperature T_m, the governing equations and their associated boundary conditions can be obtained from any standard heat-conduction text [1]. For an integral analysis, an overall energy balance for the whole slab can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho c_l \int_0^H T_l\,\mathrm{d}x + \rho LH + c_s \int_H^{H+\delta} T_s\,\mathrm{d}x - \rho c_s(H+\delta)T_l\right] = -k_l\left(\frac{\partial T_l}{\partial x}\right)_0 \quad (1)$$

where T is the temperature, H the location of the melting interface, α the thermal diffusivity, k the thermal conductivity,

ρ the density which is assumed to remain constant during the melt, L the latent heat of melting, c the specific heat, δ the penetration depth into the solid region and the subscripts s and l denote the solid and liquid regions respectively. In the 'collocation method', the interface energy-balance condition are written as

$$k_s\left(\frac{\partial T_s}{\partial x}\right)_H - k_l\left(\frac{\partial T_l}{\partial x}\right)_H = \frac{\rho L}{\alpha_s}\left(\frac{\partial T_s}{\partial x}\right)_H \Big/ \left(\frac{\partial^2 T_s}{\partial x^2}\right)_H, \quad (2)$$

$$k_s\left(\frac{\partial T_s}{\partial x}\right)_H - k_l\left(\frac{\partial T_s}{\partial x}\right)_H = \frac{\rho L}{\alpha_l}\left(\frac{\partial T_l}{\partial x}\right)_H \Big/ \left(\frac{\partial^2 T_l}{\partial x^2}\right)_H, \quad (3)$$

where α_s and α_l stand for the thermal diffusivity at the solid and liquid regions.

Except for the case with negligible heat capacity [6], it can be shown that the simplest approximate expressions for T_s and T_l are second degree polynomials. Consistent with the usual set of boundary conditions [1], these expressions are

$$T_l = T_m + D_1(x - H) + D_2(x - H)^2, \quad (4)$$

$$T_s = T_m - \frac{T_m}{\delta}(x - H)\left(2 - \frac{x - H}{\delta}\right). \quad (5)$$

Together with a prescribed boundary conditions at $x = 0$, equations (1)–(3) are sufficient for the solution for D_1, D_2, H and δ appearing in equations (4) and (5).

APPROXIMATE SOLUTIONS

Melting of solid with fixed boundary temperature

This is the famous Neumann problem. Introducing the following dimensionless temperature

$$\theta = \frac{c_l(T - T_i)}{L}. \quad (6)$$

Equations (1)–(15) can be solved to yield the following expression for the melting thickness H.

$$H(t) = 2\psi(\alpha_l t)^{1/2} \quad (7)$$

where

$$\psi^2 = \frac{B_1 + \alpha_{sl}}{\left[\frac{1}{2} + R + \frac{1}{3}B_1 R_\alpha \frac{k_{sl}}{\alpha_{sl}} + \frac{1}{St}\right](\alpha_{sl} + 2B_1) - \frac{1}{6}\alpha_{sl}}, \quad (8)$$

with

$$B_1 = k_{sl}R + \frac{\alpha_{sl}}{2St} + \left[\left(k_{sl}R + \frac{\alpha_{sl}}{2St}\right)^2 + k_{sl}\alpha_{sl}R + \frac{\alpha_{sl}^2}{2St}\right]^{1/2}, \quad (8a)$$

$$\alpha_{sl} = \frac{\alpha_s}{\alpha_l}, \quad (8b)$$

$$k_{sl} = \frac{k_s}{k_l} \quad (8c)$$

and $R = \theta_m/(\theta_w - \theta_m)$ and $St = \theta_w - \theta_m$ stand for the dimensionless subcooling ratio and Stefan number respectively. The accuracy of equation (7) can be readily established by direct comparison with the exact solution available in references [1] and [5]. In all cases, the maximum error is less than 10%.

Melting of solid with prescribed heat flux at the boundary

Introducing the following expressions for the dimensionless penetration depth and melting thickness

$$\phi = \frac{F\delta}{\rho L\alpha_l}, \quad \sigma = \frac{FH}{\rho L\alpha_l} \quad (9)$$

where F is the constant heat flux applied at the boundary $x = 0$, solution to equations (1)–(5) yields

$$\tfrac{1}{6}\alpha_{sl}\sigma^3 + \sigma^2\left[\tfrac{1}{2}\phi + \alpha_{sl}(1 + \theta_m)\right] + \sigma\left[(1 + \theta_m)\phi + \tfrac{1}{3}k_{sl}\theta_m(\phi - B_2)\right] + \tfrac{1}{3}\left(\frac{k_{sl}}{\alpha_{sl}}\right)\theta_m\phi(\phi - B_2) = \tau[\phi + \alpha_{sl}\sigma] \quad (10)$$

where B_2 is a constant given by

$$B_2 = 2k_{sl}\theta_m + \alpha_{sl} \quad (11)$$

and τ is a dimensionless time defined by

$$\tau = \frac{F(t)}{\rho^2 L^2 \alpha_l}\int_{t_0}^{t} F(t')\,dt'. \quad (12)$$

In the above expression, t_0 is the time when melting begins. Physically, the temperature–time history on the boundary is of practical interest and the result is

$$\theta_w = \theta_m + \frac{2\phi\sigma + \alpha_{sl}\sigma^2}{2\phi + 2\alpha_{sl}\sigma}. \quad (13)$$

The exact solution for the melting problem with prescribed heat flux boundary condition is not yet available. The accuracy of equations (16) and (13) must, therefore, be established by comparison with a numerical solution. Utilizing a recently-developed variable-time-step (VTS) numerical technique [7], accurate numerical results for σ and θ_w can be readily generated. Results show that the accuracy of equations (10) and (13) are excellent. It is important to note that as in the case with no initial subcooling [2], the present heat-balance integral analysis is accurate only when $F(t)$ is monotonically increasing or constant. If $F(t)$ is a pulse-type function, equation (1) generally is not accurate and alternative methods must be applied.

Melting of solid with convective heat flux at the boundary

For $t > t_0$ the appropriate dimensionless temperature, penetration depth and melting thickness for this problem are

$$\omega = \frac{h^2(t - t_0)}{\rho c_l k_l}, \quad \lambda = \frac{h\delta}{k_l}, \quad \beta = \frac{hH}{k_l}. \quad (14)$$

Equation (1) is reduced to

$$\frac{d}{d\omega}\left[G + \beta + \tfrac{1}{3}\frac{k_{sl}}{\alpha_{sl}}\theta_m\lambda\right] = \frac{2(\theta_\infty - \theta_m)(\lambda + \alpha_{sl})}{2\lambda(1 + \beta) + a_{sl}(2\beta + \beta^2)} \quad (15)$$

where

$$G = \frac{\theta_m[2\lambda\beta(1 + \beta) + \alpha_{sl}\beta^2(2 + \beta)] + (\theta_\infty - \theta_m)[\lambda\beta^2 + \tfrac{1}{3}\alpha_{sl}\beta^3]}{2\lambda(1 + \beta)} \quad (16)$$

While equation (15) cannot be integrated analytically in closed form, β can be evaluated by a simple numerical integration. Once β is obtained, the wall temperature is given by

$$\theta_w = \frac{\theta_m[2\lambda(1 + \beta) + 2\beta + 5\beta^2 + \beta^3 + 2\alpha_{sl}\beta] + (\theta_\infty - \theta_m)[2\beta + 5\beta^2 + \beta^3]}{(1 + \beta)[2\lambda(1 + \beta) + \alpha_{sl}(2\beta + \beta^2)]}. \quad (17)$$

Comparison between the value of β obtained from equation (15) and the numerical result generated by the VTS method for various values of θ_m and $\theta_\infty - \theta_m$ are conducted. The agreement is excellent for all cases. Equation (17) is expected to have the same degree of accuracy.

RESULTS AND DISCUSSION

As expected, results show that initial subcooling has two major effects on the melting process. For cases with prescribed heat flux and convective heat-transfer boundary

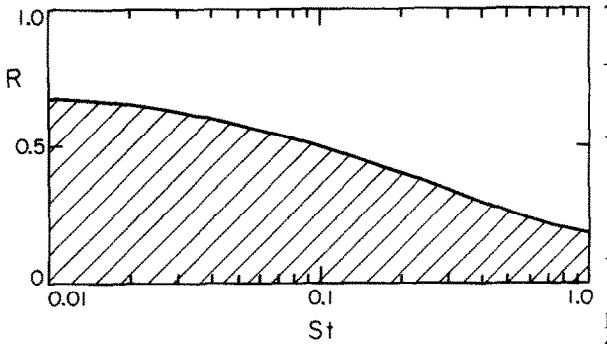

FIG. 1. Region of applicability of the no-subcooling assumption for the melting problem with constant wall temperature. The shaded region represents values of R and St such that $1 - \psi(R, St)/\psi(0, St) \leq 0.1$. ($\alpha_{sl} = k_{sl} = 1.0$).

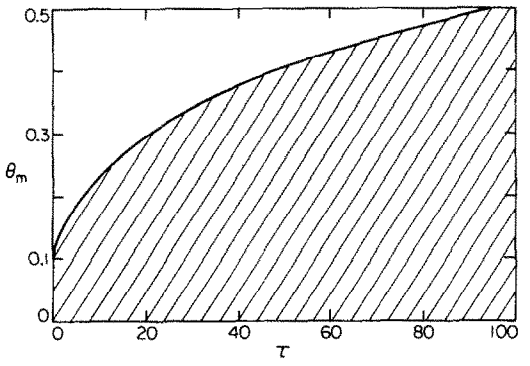

FIG. 2. Region of applicability of the no-subcooling assumption for the melting problem with a prescribed heat flux at the wall. The shaded region represents values of θ_m and τ such that $1 - \sigma(\theta_m, \tau)/\sigma(0, \tau) \leq 0.1$. ($\alpha_{sl} = k_{sl} = 1.0$).

conditions, it delays the start of the melting until the temperature of the boundary reaches the melting temperature. In all three cases, initial subcooling slows down the rate of propagation of the melting interface.

Until recently [5, 7], most analysis of phase-change problems considered only cases with no initial subcooling. Mathematically, this represents a substantial simplification. It was generally felt that by the time the melting temperature is reached on the boundary, the solid would have been heated to such an extent that it will be virtually at the melting temperature at the neighborhood of the melting interface. Despite its immense popularity, it is interesting to note that this assumption has never been justified either analytically or numerically. Only a limited mathematical discussion of this point, for example, is given by Evans *et al.* [8]. The present results provide a good basis for a quantitative evaluation of this important assumption.

For the case with constant wall temperature, it can be shown that at a given value of St, the deviation of the melting constant ψ with $R \neq 0$ from that with $R = 0$ can be quite substantial. Based on equation (18) and assuming $\alpha_{sl} = k_{sl} = 1.0$, the values of St and R such that the no-subcooling result ($R = 0$) constitutes an adequate approximation (less than 10% relative error) for the actual subcooling result ($R \neq 0$) are tabulated and presented as the shaded region in Fig. 1. The relatively small area of the shaded region indicates that in most cases the effect of subcooling is quite significant.

For the case with a prescribed heat flux at the boundary, subcooling also has a noticeable effect on the melting process.

Table 1. Effects of α_{sl} and k_{sl} on the melting constant ψ with $St = 1.0$ and $R = 1.0$

α_{sl} / k_{sl}	1.0	2.0	3.0	4.0
1.0	0.3943	0.4229	0.4349	0.4417
2.0	0.2975	0.3437	0.3657	0.3789
3.0	0.2309	0.2818	0.3086	0.3256
4.0	0.1863	0.2356	0.2637	0.2925

During the early stage of melting ($\tau \to 0$), equation (10) shows that except for small θ_m, the non-subcooling result will not be an adequate approximation for the subcooling cases. At the later stage of melting, on the other hand, the non-subcooling assumption is quite acceptable. Figure 2 illustrates this conclusion more quantitatively for cases with $\alpha_{sl} = k_{sl} = 1.0$. Based on equation (10), the values of θ_m and τ such that the no-subcooling result ($\theta_m = 0$) differ from the subcooling result ($\theta_m \neq 0$) by less than 10% relative error are tabulated and presented as the shaded region in that figure. Since τ is a dimensionless time inversely proportional to the heating rate F, Fig. 2 agrees essentially with the traditional assumption that for a slow heating rate (and therefore a large τ), the effect of initial subcooling is unimportant. It is interesting to note, however, that for all cases with moderate or large values of subcooling ($\theta_m > 0.0833$) and independent of the heating rate F, the non-subcooling result always differs significantly from the corresponding subcooling cases during the initial stage of melting.

For the melting problem with a convective heat-flux boundary condition, the effect of initial subcooling is even more significant than the previous two cases. At the early stage of melting, the relative difference between the non-subcooling solution and the subcooling result is quite large except for very small degree of initial subcooling ($\theta_m \to 0$). At the later stage of melting ($\omega \to \infty$), unlike the case with the prescribed heat-flux boundary condition in which the non-subcooling result is always accurate in the limit of large melting time, there are values of $\theta_\infty - \theta_m$ and θ_m such that at all times, the non-subcooling solution is not an accurate approximation for the corresponding subcooling case.

Finally, results of the present analysis also show the important effect of thermal property ratios α_{sl} and k_{sl} on the melting process. Without initial subcooling, α_{sl} and k_{sl} have no effect on the melting result since the analysis does not involve the solid region. With initial subcooling, however, the effect of α_{sl} and k_{sl} can be quite substantial. Table 1 illustrates some typical results on the influence of α_{sl} and k_{sl} on the melting thickness. Assuming $R = 1.0$ and $St = 1.0$, the melting constant ψ based on equation (18) is tabulated for different values of α_{sl} and k_{sl}. Generally, the melting rate increases with increasing α_{sl} and decreasing k_{sl}. Physically, it is important to note that large values of α_{sl} and k_{sl} are not unusual among common materials. For water and ice, for example, α_{sl} and k_{sl} are 7.99 and 3.68 respectively.

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SELF-PRESERVATION OF SLIGHTLY HEATED SMALL PERTURBATION JETS AND WAKES IN A PRESSURE GRADIENT

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NOMENCLATURE

C_1, C_2, C_3, C_4, C_5, constants defined in text;
$e(\eta)$, $e_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
$f(\eta)$, $f_\theta(\zeta)$, self-preserving functions defined by equations (2) and (22);
$g(\eta)$, $g_\theta(\zeta)$, self-preserving functions defined by equations (3) and (23);
$h(\eta)$, $h_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
$k(\eta)$, $k_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
l_0, l_θ, scaling lengths for the velocity and the temperature, respectively;
m, exponent for the x variation of U_1, equation (15);
n, exponent for the x variation of u_0, equation (16);
p, exponent for the x variation of l_0, equation (16); also kinematic fluctuating pressure, equation (9);
q^2, $u^2 + v^2 + w^2$;
T, mean temperature;
U, V, mean velocities in the x and y directions;
U_1, free stream velocity;
u, v, w, fluctuating velocities in the x, y and z directions;
u_0, velocity scale;
x, axial distance;
y, distance normal to axis of symmetry.

Greek symbols

α, thermal diffusivity of fluid;
α_1, β, γ, exponential indices defined in text;
$\varepsilon, \varepsilon_\theta$, mean dissipation of turbulent energy, equation (9), and of temperature, equation (31);
η, y/l_0;
ζ, y/l_θ;
θ_0, temperature scale;
ν, kinematic viscosity of fluid.

Others

prime, denotes derivative with respect to the argument of the function;
overbar, denotes time average.

GARTSHORE and Newman [1] have shown that the approximately self-preserving isothermal jet or wake in zero pressure gradient is only a particular example of a class of approximately self-preserving flows in pressure gradients. Necessary conditions for self-preservation in both two-dimensional and axisymmetric flows were obtained from the mean flow momentum equation and the turbulent energy equation. Townsend [2] showed that if small amounts of heat are present in a turbulent flow that is developing in self-preserving fashion, the temperature distribution may also be of self-preserving form. In particular, when the velocity increment in the case of a jet or the deficit in the case of a wake is proportional to the velocity U_1 of the streaming flow, the development is possible only if the temperature scale θ_0 is proportional to the velocity increment (or deficit). In this note, it is shown that the temperature length scale is proportional to the velocity length scale l_0 for the case of a slightly heated two-dimensional (or axisymmetric) jet and wake in which the velocity increment or deficit is small compared with U_1. Also, the streamwise variation of θ_0 is obtained and the condition for exponential variation of θ_0 will be made precise. With $U_1 \sim x^m$, the bounds on m for a two-dimensional flow are given [1] by $-1/3 \leq m \leq 0$. In particular, when $m = -1/3$, the length scale l_0 varies linearly, the velocity scale U_1 varies as $x^{-1/3}$ and θ_0 varies as $x^{-2/3}$. In a zero pressure gradient (e.g. Newman [3]), $l_0 \propto x^{1/2}$, $u_0 \propto x^{-1/2}$. It is shown here that, for the latter case, θ_0 is proportional to u_0.

The mean momentum equation for a two-dimensional flow can be approximated to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial \overline{uv}}{\partial y} = U_1\frac{dU_1}{dx} + \nu\frac{\partial^2 U}{\partial y^2}. \quad (1)$$

Using Townsend's [2] notation, the self-preservation forms for the velocity field are assumed to be given by

$$U = U_1 + u_0 f(\eta), \quad (2)$$

$$\overline{uv} = u_0^2 g(\eta). \quad (3)$$

The normal velocity V is obtained by integrating the continuity equation (assuming constant density), viz.

$$V = -l_0\frac{dU_1}{dx}\eta + u_0\frac{dl_0}{dx}\eta f - \frac{d}{dx}(u_0 l_0)\int_0^\eta f\,d\eta. \quad (4)$$

For small perturbation jets and wakes, i.e. when $|u_0| \ll U_1$, (1) may be approximated, after substitution of (2), (3) and (4) and neglecting terms of order $O((u_0/U_1)^2)$, to

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